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Zamolodchikov's tetrahedron equation and hidden structure of quantum groups

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Abstract

The tetrahedron equation is a three-dimensional generalization of the Yang–Baxter equation. Its solutions define integrable three-dimensional lattice models of statistical mechanics and quantum field theory. Their integrability is not related to the size of the lattice, therefore the same solution of the tetrahedron equation defines different integrable models for different finite periodic cubic lattices. Obviously, any such three-dimensional model can be viewed as a two-dimensional integrable model on a square lattice, where the additional third dimension is treated as an internal degree of freedom. Therefore every solution of the tetrahedron equation provides an infinite sequence of integrable 2d models differing by the size of this 'hidden third dimension'. In this paper, we construct a new solution of the tetrahedron equation, which provides in this way the two-dimensional solvable models related to finite-dimensional highest weight representations for all quantum affine algebra $U_q(\widehat{sl}(n))$, where the rank n coincides with the size of the hidden dimension. These models are related to an anisotropic deformation of the $sl(n)$-invariant Heisenberg magnets. They were extensively studied for a long time, but the hidden 3d structure was hitherto unknown. Our results lead to a remarkable exact 'rank-size' duality relation for the nested Bethe Ansatz solution for these models. Note also that the above solution of the tetrahedron equation arises in the quantization of the 'resonant three-wave scattering' model, which is a well-known integrable classical system in 2 + 1 dimensions.

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1. Introduction

The *tetrahedron equation* (TE) [1] is the three-dimensional analogue of the Yang–Baxter equation. It implies the commutativity of layer-to-layer transfer matrices [2] for three-dimensional lattice models of statistical mechanics and field theory and, thus, generalizes the most fundamental integrability structure of exactly solvable models in two dimensions [3].

The first solution of the TE was proposed by Zamolodchikov [1] and subsequently proven by Baxter [4]. Later, Baxter [5] exactly calculated the free energy of the corresponding solvable three-dimensional model in the limit of an infinite lattice. Next, Bazhanov and Baxter [6] generalized this model for an arbitrary number of spin states, N (for the original Zamolodchikov model $N = 2$). The corresponding solutions of the TE were found by Kashaev, Mangazeev, Sergeev and Stroganov [7, 8]. The other known solutions, previously found by Hietarinta [9] and Korepanov [10], were shown to be special cases of [8]. A review of the most recent activity related to the TE can be found in the work of von Gehlen, Pakulyak and Sergeev [11].

It is worth mentioning that the generalized Zamolodchikov model of [6] also possesses a much simpler integrability condition—the 'restricted star-triangle relation', introduced in [12]. Remarkably, the very same relation serves as the five-term (or the 'pentagon') identity for the quantum dilogarithm of Faddeev and Kashaev [13] and has been used by Kashaev to formulate his famous 'hyperbolic volume conjecture' [14, 15]. A review of related mathematical identities from the point of view of the theory of basic q-hypergeometric series was given by Au-Yang and Perk [16] (see also [8, 17]).

As shown in [6] the generalized N-state Zamolodchikov model has a profound connection to the theory of quantum groups [18, 19], namely to the cyclic representations of the affine quantum group $U_q(\widehat{sl}(n))$ at roots of unity, $q^N = 1$. For the cubic lattice with n layers in one direction this model is equivalent to the two-dimensional $sl(n)$-generalized chiral Potts model[1] [26, 27] (*modulo* a minor modification of the boundary conditions). In other words, this particular three-dimensional model can be viewed as a two-dimensional integrable model, where the third dimension of the lattice becomes the rank of the underlying affine quantum group [6] (for related discussions see also [28, 29]).

In this paper, we argue that the above 'three-dimensional structure' is a *generic property* of the affine quantum groups. It is neither restricted to the root of unity case, nor related to the properties of cyclic representations. To support this statement we present a new solution of the TE, such that the associated three-dimensional models reproduce the two-dimensional solvable models related to finite-dimensional highest weight representations for all the affine quantum groups $U_q(\widehat{sl}(n))$ with generic values of q. These models were discovered in the early 1980s [30–33] and have since found numerous applications in integrable systems. They are related to an anisotropic deformation of the $sl(n)$-invariant Heisenberg magnets [34–37]. In the two-layer case ($n = 2$), these models include the most general six-vertex model [38] and all its higher-spin descendants. Our approach is briefly outlined below.

It is well known that the Yang–Baxter equation

$$R_{ab}R_{ac}R_{bc} = R_{bc}R_{ac}R_{ab}, \tag{1}$$

can be regarded is the associativity condition for the L-operator algebra, defined by the 'RLL-relation' [39]

$$L_{1,a}L_{1,b}R_{ab} = R_{ab}L_{1,b}L_{1,a}. \tag{2}$$

In the similar way the TE (here it is written in the so-called vertex form)

$$R_{abc}R_{ade}R_{bdf}R_{cef} = R_{cef}R_{bdf}R_{ade}R_{abc}, \tag{3}$$

[1] Note that even the simplest two-layer case ($n = 2$) corresponds to the famous chiral Potts model [20–23] which has attracted much attention over the last 20 years. The most remarkable recent result there is Baxter's derivation [24] of the order parameter conjecture of Albertini *et al* [25].

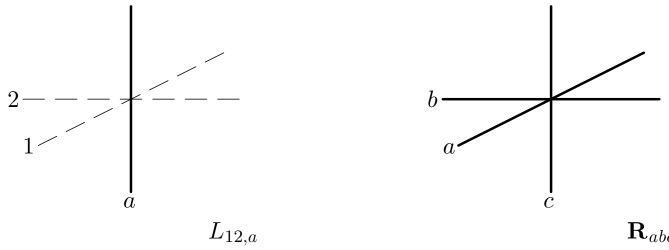

Figure 1. Graphical representation of the operators $L_{12,a}$ and R_{abc}.

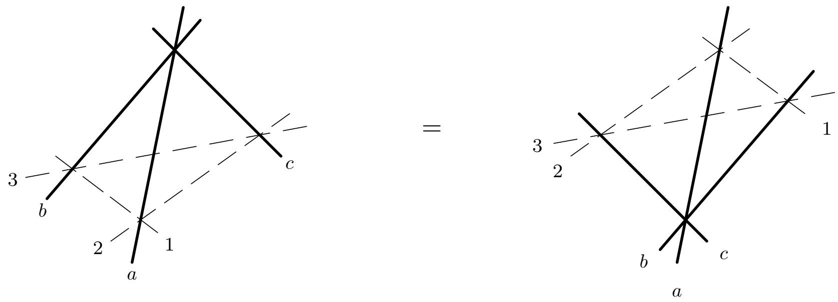

Figure 2. Graphical representation of the tetrahedron equation (4). The labels '1, 2, 3, a, b, c' on the lines indicate the corresponding vector spaces.

can be regarded as the associativity condition[2] for a three-dimensional analogue of the above 'RLL'-relation,

$$L_{12,a}L_{13,b}L_{23,c}R_{abc} = R_{abc}L_{23,c}L_{13,b}L_{12,a}. \tag{4}$$

Here all operators act in a direct product of six vector spaces $\mathcal{V} = V_1 \otimes V_2 \otimes V_3 \otimes \mathcal{F}_a \otimes \mathcal{F}_b \otimes \mathcal{F}_c$, involving the three identical 'auxiliary' vector spaces $V_i = V$, $i = 1, 2, 3$, and the three identical 'quantum' spaces $\mathcal{F}_i = \mathcal{F}$, $i = a, b, c$. The operator $L_{12,a}$ acts non-trivially only in $V_1 \otimes V_2 \otimes F_a$ and coincides with the identity operator in all the remaining components of $\mathcal{V}$. The other operators in (4) are defined similarly. The graphical representation of equation (4) is given in figures 1 and 2. Suppose next, that $\mathcal{F}$ is a representation space of some algebra $\mathcal{A}$. Obviously the space $\mathcal{F}^{\otimes 3}$ is the representation space of the tensor cube of this algebra, $\mathcal{A}_a \otimes \mathcal{A}_b \otimes \mathcal{A}_c$, where $\mathcal{A}_a = \mathcal{A} \otimes \mathrm{id} \otimes \mathrm{id}$, $\mathcal{A}_b = \mathrm{id} \otimes \mathcal{A} \otimes \mathrm{id}$ and $\mathcal{A}_c = \mathrm{id} \otimes \mathrm{id} \otimes \mathcal{A}$. Then the operator $L_{12,a}$, for instance, can be understood as an operator-valued matrix acting in $V_1 \otimes V_2$, whose elements belong to the algebra $\mathcal{A}$,

$$L_{12,a} = L_{12}(\mathbf{v}_a, s_a), \tag{5}$$

where $\mathbf{v}_a$ denotes a set of generators of $\mathcal{A}_a$ and s_a stands for a set of c-valued parameters.

With this notation equation (4) can be re-written as

$$L_{12}(\mathbf{v}_a, s_a)L_{13}(\mathbf{v}_b, s_b)L_{23}(\mathbf{v}_c, s_c) = \mathcal{R}_{abc}(L_{23}(\mathbf{v}_c, s_c)L_{13}(\mathbf{v}_b, s_b)L_{12}(\mathbf{v}_a, s_a)), \tag{6}$$

where the functional map

$$\mathcal{R}_{abc}(\phi) = R_{abc}\phi R_{abc}^{-1}, \qquad \forall \phi \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}, \tag{7}$$

[2] This associativity condition is, essentially, the Zamolodchikov's factorization condition for the scattering S-matrix of the 'straight strings' in the $2 + 1$-dimensional space [1].

defines an automorphism of the tensor cube of $\mathcal{A}$,

$$\mathcal{R}: \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}. \tag{8}$$

The relation of form (6) is sometimes called the tetrahedral Zamolodchikov algebra [10]. Since the operators R solve the tetrahedron equation (3), map (7) solves the *functional* TE

$$\mathcal{R}_{abc}(\mathcal{R}_{ade}(\mathcal{R}_{bdf}(\mathcal{R}_{cef}(\phi)))) = \mathcal{R}_{cef}(\mathcal{R}_{bdf}(\mathcal{R}_{ade}(\mathcal{R}_{abc}(\phi)))). \tag{9}$$

Further, denote the result of the action of the automorphism (7) on the basis of $\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}$ as

$$\mathbf{v}'_a = \mathcal{R}_{abc}(\mathbf{v}_a), \qquad \mathbf{v}'_b = \mathcal{R}_{abc}(\mathbf{v}_b), \qquad \mathbf{v}'_c = \mathcal{R}_{abc}(\mathbf{v}_c). \tag{10}$$

Equation (6) then takes the form of the *local Yang–Baxter equation*

$$L_{12}(\mathbf{v}_a, s_a)L_{13}(\mathbf{v}_b, s_b)L_{23}(\mathbf{v}_c, s_c) = L_{23}(\mathbf{v}'_c, s_c)L_{13}(\mathbf{v}'_b, s_b)L_{12}(\mathbf{v}'_a, s_a). \tag{11}$$

This equation was introduced by Maillet and Nijhoff [40] and further developed by Kashaev, Korepanov and Sergeev [41–43]. It is particularly useful for constructing integrable evolution systems on a 2+1-dimensional lattice and can be viewed as a three-dimensional lattice analogue of the Lax–Zakharov–Shabat zero-curvature condition in two dimensions (see [44] for further details).

Suppose now that the algebra $\mathcal{A}$ has a 'quasi-classical' limit when it degenerates into a Poisson algebra $\mathcal{P}$. In this limit all the generators $\mathbf{v}_i$, $i = a, b, c$, become commutative formal variables, and relation (11) becomes a c-number equation which very much resembles the usual quantum Yang–Baxter equation (but does not coincide with it). The quasi-classical limit of map (7) defines a symplectic transformation of the tensor cube of the Poisson algebra $\mathcal{P}$, which, of course, solves the same functional TE (9).

The approach we employ in this paper can be viewed as a 'reverse engineering' of the above procedure. We start from the c-number equation (11) with a simple particular Ansatz for the matrices L_{12}, L_{13} and L_{23} acting in the product of two-dimensional vector spaces $\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. Solving this equation we construct a symplectic transformation of the tensor cube of the Poisson algebra $\mathcal{P}$, defined by the brackets

$$\mathcal{P}: \quad \{x, y\} = 1 - xy, \qquad \{h, y\} = y, \qquad \{h, x\} = -x. \tag{12}$$

A natural quantization of this algebra leads to the q-deformed Heisenberg algebra $\mathcal{A} = \mathcal{H}_q$ (or the q-oscillator algebra [45, 46]),

$$\mathcal{H}_q: \quad q\,\mathbf{yx} - q^{-1}\mathbf{xy} = q - q^{-1}, \qquad [\mathbf{h}, \mathbf{y}] = \mathbf{y}, \qquad [\mathbf{h}, \mathbf{x}] = -\mathbf{x}. \tag{13}$$

Surprisingly, only minor modifications of the above symplectic transformation and the form of the matrices L are required to obtain the quantum map (7) for the tensor cube of $\mathcal{H}_q$, which solves the quantum variant of equation (11). Finally, we obtain the corresponding solution of the TE (3) by using an explicit form of the quantum map (7) to calculate matrix elements of the operator R_{abc} for the Fock representation of the q-oscillator algebra (13).

In this paper, we briefly present our main results, postponing the details to the future publication [44]. This paper is organized as follows. The solution of the local Yang–Baxter equation (11) with commutative variables (classical case) is given in section 2. The quantum case is considered in section 3. A new solution to the tetrahedron equations (3) and (4) is presented in section 4. The connection of this solution with certain two-dimensional solvable models and the theory of quantum group is discussed in section 5. In section 6, we derive the 'rank-size' (RS) duality for these two-dimensional models. In the conclusion, we summarize our results and discuss some open questions.

2. Local Yang–Baxter equation

Our starting point is the local Yang–Baxter equation (11), where each matrix L depends on two pairs of variables, a pair of *dynamical* variables $\mathbf{v} = (x, y)$ and a pair of (complex) parameters $s = (\lambda, \mu)$. Altogether equation (11) contains six different sets $\mathbf{v}_i, \mathbf{v}'_i$, $i = a, b, c$ (each L depends on its own set $\mathbf{v}$) and three sets of parameters s_i, $i = a, b, c$ (they remain the same for both sides of the equation).

Any matrix acting in the product $\mathbb{C}^2 \otimes \mathbb{C}^2$ can be conveniently presented as a two by two block matrix with two-dimensional blocks where the matrix indices related to the second vector space numerate the blocks while the indices of the first space numerate matrix elements inside the blocks. With this conversion define the matrix

$$L_{12}(\mathbf{v}, s): \qquad \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2, \tag{14}$$

as follows

$$L_{12}(\mathbf{v}; s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda k & y & 0 \\ 0 & -\lambda\mu x & \mu k & 0 \\ 0 & 0 & 0 & -\lambda\mu \end{pmatrix}, \tag{15}$$

where $\mathbf{v} = (x, y)$, $s = (\lambda, \mu)$ and

$$k^2 = 1 - xy, \qquad k \to 1 \qquad \text{when} \quad xy \to 0. \tag{16}$$

Now let us consider (11) as an equation for $\mathbf{v}'_a, \mathbf{v}'_b, \mathbf{v}'_c$, regarding $\mathbf{v}_a, \mathbf{v}_b, \mathbf{v}_c$ and s_a, s_b, s_c as given. It should be stressed that we are seeking for a solution where no constraints on the variables $\mathbf{v}_i, s_i$, $i = a, b, c$, on the left-hand side of (11) are imposed. This requirement excludes the case when $\mathbf{v}_i = \mathbf{v}'_i$, where equation (11) reduces to the ordinary Yang–Baxter equation for the six-vertex free-fermion model[3] (note that the matrix elements of (15) satisfy the free-fermion condition).

Detailed considerations show that equation (11) contains exactly six algebraically independent equations, which can be uniquely solved for six unknowns x'_i, y'_i, $i = a, b, c$,

$$\begin{aligned} x'_a &= k'^{-1}_b \frac{\lambda_b}{\lambda_c}\left(k_c x_a - \frac{1}{\lambda_a\mu_c} k_a x_b y_c\right), & y'_a &= k'^{-1}_b \frac{\lambda_c}{\lambda_b}(k_c y_a - \lambda_a\mu_c k_a y_b x_c), \\ x'_b &= x_a x_c + \frac{1}{\lambda_a\mu_c} k_a k_c x_b, & y'_b &= y_a y_c + \lambda_a\mu_c k_a k_c y_b, , \\ x'_c &= k'^{-1}_b \frac{\mu_b}{\mu_a}\left(k_a x_c - \frac{1}{\lambda_a\mu_c} k_c y_a x_b\right), & y'_c &= k'^{-1}_b \frac{\mu_a}{\mu_b}(k_a y_c - \lambda_a\mu_c k_c x_a y_b), \end{aligned} \tag{17}$$

where $k'^2_j = 1 - x'_j y'_j$, in particular,

$$k'_b{}^2 = k_a^2 k_b^2 k_c^2 - 2k_a^2 k_c^2 + k_a^2 + k_c^2 - \frac{k_a k_c y_a x_b y_c}{\lambda_a\mu_c} - \lambda_a\mu_c k_a k_c x_a y_b x_c. \tag{18}$$

Elements k'_a and k'_c can be found from

$$k'_a k'_b = k_a k_b, \qquad k'_b k'_c = k_b k_c. \tag{19}$$

The signs of k'_a, k'_b and k'_c are fixed by condition (16).

It is not difficult to verify by explicit calculations that the functional operator (10) associated with transformation ((17) and (18)) satisfies the functional TE (9), as it, of course,

[3] In the case of the ordinary Yang–Baxter equation the three matrices L_{12}, L_{13} and L_{23} are not independent.

should. In addition, one can verify that the same transformation preserves the Poisson structure of the tensor cube of algebra (12), where the element $k \equiv \mathrm{e}^{-h/2}$ is constrained by relation (16),

$$k^2 = 1 - xy, \qquad k^2 = \mathrm{e}^{-h}. \tag{20}$$

In other words, the set of Poisson brackets

$$\{x_j, y_j\} = 1 - x_j y_j, \qquad j = a, b, c, \tag{21}$$

(where all other brackets are zero) imply the same set of brackets for the 'primed' variables x'_j, y'_j, $j = a, b, c$.

It should be noted that the general solution of the local Yang–Baxter equation (11) with the six-vertex-type operators L_{ij}, satisfying the free-fermion condition, was found by Korepanov [42]. This solution, however, cannot be immediately understood as a transformation of variables between the left-hand and right-hand sides of (11), since it contains some spurious degrees of freedom which cancel out in each side of this equation separately. We eliminate these 'unwanted' degrees of freedom and choose a special parametrization (15) of the matrices L, such that the resulting map ((17) and (18)) preserves the Poisson structure of the tensor cube of algebra 12). The reasons why this is possible at all and why does algebra (12) appear here currently remain unclear.

3. Quantization

The simplest quantization of the Poisson algebra (12) with constraint (20) is the q-oscillator algebra (13) with the element $\mathbf{k} \equiv q^{\mathbf{h}}$ obeying the relation

$$\mathbf{k}^2 = 1 - \mathbf{y}\mathbf{x}, \qquad \mathbf{k} = q^{\mathbf{h}}. \tag{22}$$

Using the same conventions as in (15), assume the following Ansatz for the operator-valued matrices L in (11)

$$L_{12}(\mathbf{v}; s) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda\mathbf{k} & \mathbf{y} & 0 \\ 0 & -q^{-1}\lambda\mu\mathbf{x} & \mu\mathbf{k} & 0 \\ 0 & 0 & 0 & -q^{-1}\lambda\mu \end{pmatrix} \tag{23}$$

where $\mathbf{v} = (\mathbf{x}, \mathbf{y}), s = (\lambda, \mu)$ and the parameter q is the same as in (13). Obviously, equation (11) becomes an operator equation. It involves three mutually commuting sets of generators $\mathbf{x}_j, \mathbf{y}_j$, $j = a, b, c$, satisfying (13) and (22). Just as in the commuting variable case of section 2, one can solve (11) for operators $\mathbf{x}'_j, \mathbf{y}'_j$, $j = a, b, c$. Note that no assumptions on their commutation properties are required. The solution is

$$\begin{aligned}
\mathbf{x}'_a &= \mathbf{k}'^{-1}_b \frac{\lambda_b}{\lambda_c}\left(\mathbf{k}_c\mathbf{x}_a - \frac{q}{\lambda_a\mu_c}\mathbf{k}_a\mathbf{x}_b\mathbf{y}_c\right), & \mathbf{y}'_a &= \mathbf{k}'^{-1}_b \frac{\lambda_c}{\lambda_b}\left(\mathbf{k}_c\mathbf{y}_a - \frac{\lambda_a\mu_c}{q}\mathbf{k}_a\mathbf{y}_b\mathbf{x}_c\right), \\
\mathbf{x}'_b &= \mathbf{x}_a\mathbf{x}_c + \frac{q^2}{\lambda_a\mu_c}\mathbf{k}_a\mathbf{k}_c\mathbf{x}_b, & \mathbf{y}'_b &= \mathbf{y}_a\mathbf{y}_c + \lambda_a\mu_c\mathbf{k}_a\mathbf{k}_c\mathbf{y}_b, \\
\mathbf{x}'_c &= \mathbf{k}'^{-1}_b \frac{\mu_b}{\mu_a}\left(\mathbf{k}_a\mathbf{x}_c - \frac{q}{\lambda_a\mu_c}\mathbf{k}_c\mathbf{y}_a\mathbf{x}_b\right), & \mathbf{y}'_c &= \mathbf{k}'^{-1}_b \frac{\mu_a}{\mu_b}\left(\mathbf{k}_a\mathbf{y}_c - \frac{\lambda_a\mu_c}{q}\mathbf{k}_c\mathbf{x}_a\mathbf{y}_b\right),
\end{aligned} \tag{24}$$

where

$$\mathbf{k}'^2_b = q^2\mathbf{k}^2_a\mathbf{k}^2_b\mathbf{k}^2_c - (1+q^2)\mathbf{k}^2_a\mathbf{k}^2_c + \mathbf{k}^2_a + \mathbf{k}^2_c - \frac{\mathbf{k}_a\mathbf{k}_c\mathbf{y}_a\mathbf{x}_b\mathbf{y}_c}{\lambda_a\mu_c} - \lambda_a\mu_c\mathbf{k}_a\mathbf{k}_c\mathbf{x}_a\mathbf{y}_b\mathbf{x}_c \tag{25}$$

and

$$\mathbf{k}'_a\mathbf{k}'_b = \mathbf{k}_a\mathbf{k}_b, \qquad \mathbf{k}'_b\mathbf{k}'_c = \mathbf{k}_b\mathbf{k}_c. \tag{26}$$

Again one can verify directly that this operator transformation is an automorphism of the tensor cube of q-oscillator algebra ((13) and (22)), and that the functional operator (10) corresponding to this transformation solves the quantum functional TE (9) as well.

4. A solution of the tetrahedron equation

Consider the Fock representation, $\mathcal{F}$, of the q-oscillator algebra (13) spanned on the basis $|\alpha\rangle$, $\alpha = 0, 1, 2, \ldots, \infty$,

$$\mathcal{F}: \quad \mathbf{x}|0\rangle = 0, \qquad \mathbf{y}|\alpha\rangle = (1 - q^{2\alpha+2})|\alpha + 1\rangle, \qquad \mathbf{h}|\alpha\rangle = \alpha|\alpha\rangle, \qquad \langle\alpha|\alpha'\rangle = \delta_{\alpha,\alpha'} \tag{27}$$

and assume that q is not a root of unity. Then representation (27) is irreducible and therefore equation (7) determines the matrix elements of the operator R_{abc} to within an unessential scale factor. Note that constraint (20) for the representation (27) is fulfilled automatically.

Explicitly, equation (7) reads

$$\mathbf{x}'_j R_{abc} = R_{abc}\mathbf{x}_j, \qquad \mathbf{y}'_j R_{abc} = R_{abc}\mathbf{y}_j, \qquad j = a, b, c, \tag{28}$$

where $\mathbf{x}'_j, \mathbf{y}'_j$ are given by (24). Solving these equations one obtains

$$\langle\alpha, \beta, \gamma|R_{abc}|\alpha', \beta', \gamma'\rangle = (\lambda_c/\lambda_b)^{\alpha'} (\lambda_a\mu_c)^{\beta} (\mu_a/\mu_b)^{\gamma'} \langle\alpha, \beta, \gamma|\mathtt{r}|\alpha', \beta', \gamma'\rangle \tag{29}$$

where matrix indices, (α, α'), (β, β') and (γ, γ'), refer to the representation spaces $\mathcal{F}_a, \mathcal{F}_b$ and $\mathcal{F}_c$, respectively, and take the values $0, 1, 2, \ldots, \infty$. Note that the parameters $s_j = (\lambda_j, \mu_j)$, $j = a, b, c$, associated with these spaces in (11), enter (29) only through simple power factors. The constant matrix $\langle\ldots|\mathtt{r}|\ldots\rangle$ is given by

$$\langle\alpha, \beta, \gamma|\mathtt{r}|\alpha', \beta', \gamma'\rangle = (-)^{\beta}\delta_{\alpha+\beta,\alpha'+\beta'}\delta_{\beta+\gamma,\beta'+\gamma'}\frac{q^{(\alpha'-\beta)(\gamma'-\beta)-\beta}}{(q^2; q^2)_{\beta'}}P_{\beta'}(q^{2\alpha}, q^{2\beta}, q^{2\gamma}), \tag{30}$$

where

$$(x; q^2)_n = \prod_{j=0}^{n-1}(1 - q^{2j}x).$$

The polynomials $P_n(x, y, z)$ are defined recursively,

$$P_n(x, y, z) = (1 - x)(1 - z)P_{n-1}(q^{-2}x, y, q^{-2}z) - q^{2-2n}xz(1 - y)P_{n-1}(x, q^{-2}y, z), \tag{31}$$

with the initial condition $P_0(x, y, z) = 1$. The first non-trivial polynomial reads

$$P_1(x, y, z) = 1 - x - z + xyz, \tag{32}$$

whereas all higher polynomials soon become very complicated and their explicit form is not very illuminating[4].

Recurrence (31) ensures that all the defining relations (28) for the operators R_{abc} (which are equivalent to the TE (4) are satisfied. Thus the operators L and R_{abc} defined by (5), (23) and (29), respectively, solve the TE (4). Further, the general arguments of section 1 and the irreducibility of the representation (27) imply that the operators R_{abc}, defined above, also solve the TE (3).

Note that all the parameters λ_j and μ_j associated with six different vector spaces labelled by a, b, c, d, e and f totally cancel out from equation (3). Therefore, this equation is essentially a *constant* TE equation for the parameter-independent matrix $\mathtt{r}$, (30),

$$\mathtt{r}_{abc}\mathtt{r}_{ade}\mathtt{r}_{bdf}\mathtt{r}_{cef} = \mathtt{r}_{cef}\mathtt{r}_{bdf}\mathtt{r}_{ade}\mathtt{r}_{abc}, \tag{33}$$

[4] Nevertheless, Mangazeev [47] found an elegant closed form solution for these polynomials in terms of the hypergeometric function, which will be reported elsewhere [48].

which coincides with equation (3) when all the λ's and μ's therein are equal to one. The same remark applies to the other TE (4) (and also to equation (34)) as well. Nevertheless, the above parameters play an important role for the associated two-dimensional R-matrices considered in the next section and that is why they are retained here. Note also that due to the presence of two delta functions in (30) each side of (3) contains only a finite summation over the intermediate matrix indices.

There exists another relevant TE, which we present without a proof. It involves by the same operators L,

$$M_{0\bar{0},a}(\mu/\mu')M_{1\bar{1},a}(\lambda'/\lambda)L_{01,b}(s)L_{\bar{0}\bar{1},b}(s') = L_{\bar{0}\bar{1},b}(s')L_{01,b}(s)M_{1\bar{1},a}(\lambda'/\lambda)M_{0\bar{0},a}(\mu/\mu') \tag{34}$$

and new operators $M_{ij,a}$ which are very much similar to L's. They act non-trivially in the product of three spaces $V_i \otimes V_j \otimes \mathcal{F}_a$, where $V_i = V_j = \mathbb{C}^2$ and $\mathcal{F}_a$ is the Fock space (27). With the same conventions as in (15) these operators are defined as

$$M_{12,a}(\lambda) = \begin{pmatrix} \lambda^{\mathbf{h}} & 0 & 0 & 0 \\ 0 & \bar{\mathbf{k}}\lambda^{\mathbf{h}} & \mathbf{y}\lambda^{\mathbf{h}} & 0 \\ 0 & q^{-1}\mathbf{x}\lambda^{\mathbf{h}} & \bar{\mathbf{k}}\lambda^{\mathbf{h}} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda^{\mathbf{h}} \end{pmatrix}, \qquad \bar{\mathbf{k}} = (-q)^{\mathbf{h}}, \tag{35}$$

where $\mathbf{x}$, $\mathbf{y}$, $\mathbf{h}$ are the generators of q-oscillator algebra (13) (recall that the element $\mathbf{h}$ has the integer-valued spectrum (27), so the power $\lambda^{\mathbf{h}}$ is well defined). The proof of (34) is given in [44].

5. Quantum R-matrices for $U_q(\widehat{sl}(n))$

Consider a model of statistical mechanics on a cubic lattice with the toroidal boundary conditions in all three lattice directions. Let each edge carry a discrete spin variable taking the values $0, 1, 2, \ldots, \infty$ and each vertex is assigned with Boltzmann weights, given by the matrix R_{abc}, (29), so that its indices are identified with the spin variables on six edges surrounding the vertex. Namely, the three pairs of indices (α, α'), (β, β') and (γ, γ') are associated with the edges oriented along three different lattice directions 'a', 'b' and 'c' respectively.

It is well known that any edge-spin model on the cubic model can be viewed as a two-dimensional model on the square lattice with an enlarged space of states for the edge spins (see [2, 6] for additional explanations). Consider the following quantity,

$$\mathbf{R_{bc}} = \mathrm{Tr}_{\mathcal{F}_a}\big(R_{ab_1c_1}R_{ab_2c_2}\cdots R_{ab_nc_n}\big), \tag{36}$$

which involves the product of the vertex weights along a whole line of vertices in the direction 'a' of the lattice, with n being the corresponding lattice size (see figure 3). It is an operator acting in the product of two 'composite' spaces $\mathbf{F_b} \otimes \mathbf{F_c}$,

$$\mathbf{F_b} = \mathcal{F}_{b_1} \otimes \mathcal{F}_{b_2} \otimes \cdots \otimes \mathcal{F}_{b_n}, \qquad \mathbf{F_c} = \mathcal{F}_{c_1} \otimes \mathcal{F}_{c_2} \otimes \cdots \otimes \mathcal{F}_{c_n}, \tag{37}$$

where each space $\mathcal{F}$ coincides with the Fock space (27). It is easy to show that operator (36) satisfies the Yang–Baxter equation,

$$\mathbf{R_{bc}}\mathbf{R_{bd}}\mathbf{R_{cd}} = \mathbf{R_{cd}}\mathbf{R_{bd}}\mathbf{R_{bc}} \tag{38}$$

as a simple consequence of the TE (3) and the fact that the matrix (30) is a non-degenerate matrix in $\mathcal{F}^{\otimes 3}$. Evidently, operator (36) is the R-matrix of the associated two-dimensional model. Similarly, one can construct the L-operator

$$\mathbf{L_{Vb}} = \mathrm{Tr}_{V_0}\big(L_{01,b_1}L_{02,b_2}\cdots L_{0n,b_n}\big), \tag{39}$$

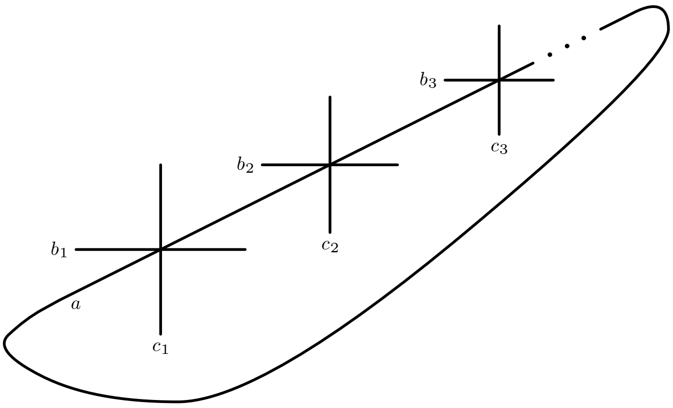

Figure 3. A chain of the operators R traced in the space $\mathcal{F}_a$.

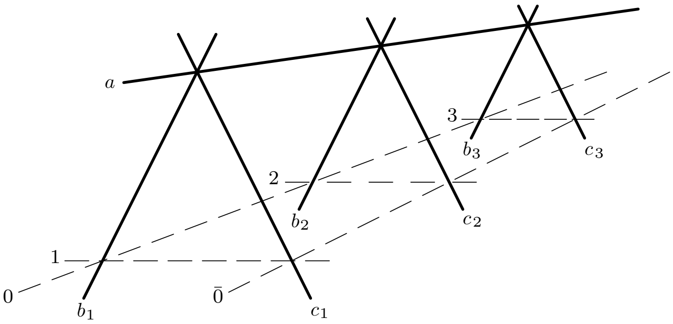

Figure 4. The local 3d structure of the product $\mathbf{L_{Vb}L_{Vc}R_{bc}}$ on the left-hand side of the Yang–Baxter equation (40).

where $L_{ij,b}$ is given by (15) and trace is taken over the two-dimensional space $V_0 = \mathbb{C}^2$. Equation (4) implies the Yang–Baxter equation

$$\mathbf{L_{Vb}L_{Vc}R_{bc}} = \mathbf{R_{bc}L_{Vc}L_{Vb}}. \tag{40}$$

The local 3d structure of this equation is shown in figure 4. Evidently, the R-matrix (36) 'intertwines' the L-operators (39) in the quantum spaces. There exists another R-matrix, $\overline{\mathbf{R}}_{\mathbf{VV'}}$, which intertwines the same operators in the auxiliary spaces

$$\mathbf{L_{Vb}L_{V'b}}\overline{\mathbf{R}}_{\mathbf{VV'}} = \overline{\mathbf{R}}_{\mathbf{VV'}}\mathbf{L_{V'b}L_{Vb}}. \tag{41}$$

This Yang–Baxter equation follows from (34).

In general, each operator L_{0j,b_j} in (39) depends on its own set of parameters $s_j = (\lambda_j, \mu_j)$. Consider the case when all λ_j are the same, $\lambda_j \equiv \lambda$, $j = 1, 2, \ldots, n$, whereas the parameters $\{\mu\} = \{\mu_1, \mu_2, \ldots, \mu_n\}$ are kept arbitrary; we will write operator (39) as $\mathbf{L_{Vb}}(\lambda|\{\mu\})$ to indicate these arguments explicitly. This operator acts in product $\mathbf{V} \otimes \mathbf{F_b}$, where

$$\mathbf{V} = (\mathbb{C}^2)^{\otimes n} \tag{42}$$

and $\mathbf{F_b}$ is defined in (37). It can be regarded as a 2^n by 2^n matrix with operator-valued entries acting in the quantum space $\mathbf{F_b}$. It turns out that this matrix has a block-diagonal structure, with $(n+1)$ blocks of the dimensions, which coincide with dimensions of the fundamental representations π_{ω_k} of the algebra $U_q(sl(n))$,

$$\dim(\pi_{\omega_k}) = \frac{n!}{k!(n-k)!}, \qquad k = 0, 1, \ldots, n. \tag{43}$$

Here ω_k denotes the highest weight of the kth fundamental representation. Note that π_{ω_0} and π_{ω_n} are the trivial one-dimensional representations. Consider, for instance, the n-dimensional subspace of (42) with the basis

$$|\psi_i\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \begin{pmatrix}1\\0\end{pmatrix} \otimes \underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{i\text{th place}} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix}, \qquad i = 1, 2, \ldots, n. \tag{44}$$

One can easily see that operator (39) acts invariantly in this subspace. The corresponding n by n matrix reads

$$\langle\psi_i|\mathbf{L}_{\mathbf{Vb}}(\lambda|\{\mu\})|\psi_j\rangle = \mathcal{L}_{ij}(\lambda|\{\mu\}) = -q^{-1}\mu_i\mathcal{L}_{ij}(\lambda), \tag{45}$$

where $\mathcal{L}_{ij}(\lambda)$ does not depend on the parameters $\{\mu\}$,

$$\mathcal{L}(\lambda) = \sum_{\alpha=1}^{n} E_{\alpha\alpha} \otimes (\lambda^n q^{\mathcal{J}-\mathbf{h}_\alpha} - q^{\mathbf{h}_\alpha}) + \sum_{\alpha<\beta}(\lambda^{\beta-\alpha}E_{\alpha\beta} \otimes \mathcal{E}_{\beta\alpha} + \lambda^{n+\alpha-\beta}E_{\beta\alpha} \otimes q^{\mathcal{J}}\mathcal{E}_{\alpha\beta}). \tag{46}$$

Here $(E_{\alpha\beta})_{i,j} = \delta_{i,\alpha}\delta_{j,\beta}$ is the standard matrix unit and

$$\mathcal{J} = \sum_{\gamma=1}^{n} \mathbf{h}_\gamma, \qquad \mathcal{E}_{\alpha\beta} = \mathbf{x}_\beta\mathbf{y}_\alpha \prod_{\gamma=\alpha}^{\beta} q^{-\mathbf{h}_\gamma}, \qquad \mathcal{E}_{\beta\alpha} = \mathbf{x}_\alpha\mathbf{y}_\beta \prod_{\gamma=\alpha+1}^{\beta-1} q^{\mathbf{h}_\gamma}, \qquad \alpha < \beta, \tag{47}$$

where $(\mathbf{x}_\alpha, \mathbf{y}_\alpha, \mathbf{h}_\alpha)$, $\alpha = 1, 2, \ldots, n$ are the generators of the n independent copies of the q-oscillator algebra (13). A very simple inspection shows that the matrix $\mathcal{L}(\lambda)$, given by (46), is nothing but the fundamental L-operator [19, 49] for the quantum affine Lie algebra $U_q(\widehat{sl}(n))$ where the parameter λ is just the usual spectral parameter. Recall that the spectral parameter dependent L-operators arise as specializations of the universal R-matrix [18, 50, 51] when the affine algebra $U_q(\widehat{sl}(n))$ is realized by means of the *evaluation homomorphism* [19] into the finite-dimensional algebra $U_q(sl(n))$. A specific feature of the L-operator (46) is that the latter algebra is realized in this case with the help of another homomorphism into the tensor power of the q-oscillator algebra (13).

The Cartan–Weyl generators $E_\alpha, F_\alpha, H_\alpha$, of the algebra $U_q(sl(n))$, where $\alpha = 1, 2, \ldots n-1$, satisfy the commutation relations

$$[H_\alpha, E_\beta] = A_{\alpha,\beta}E_\beta, \qquad [H_\alpha, F_\beta] = -A_{\alpha,\beta}F_\beta, \qquad [E_\alpha, F_\beta] = \delta_{\alpha,\beta}\frac{q^{H_\alpha} - q^{-H_\alpha}}{q - q^{-1}} \tag{48}$$

and the cubic Serre relations

$$\begin{aligned} &E_\alpha^2 E_{\alpha\pm1} - (q+q^{-1})E_\alpha E_{\alpha\pm1}E_\alpha + E_{\alpha\pm1}E_\alpha^2 = 0,\\ &F_\alpha^2 F_{\alpha\pm1} - (q+q^{-1})F_\alpha F_{\alpha\pm1}F_\alpha + F_{\alpha\pm1}F_\alpha^2 = 0, \end{aligned} \tag{49}$$

where $A_{\alpha\beta} = 2\delta_{\alpha,\beta} - \delta_{\alpha+1,\beta} - \delta_{\alpha,\beta+1}$ is the Cartan matrix. The L-operator (46) corresponds to the following well-known realization of these defining relations in terms of the q-oscillators [45, 46],

$$\begin{aligned} E_\alpha &= \frac{1}{(q-q^{-1})}q^{\frac{\mathbf{h}_{\alpha+1}}{2}}\mathcal{E}_{\alpha,\alpha+1}q^{\frac{\mathbf{h}_\alpha}{2}} = \frac{1}{(q-q^{-1})}q^{-\frac{\mathbf{h}_\alpha}{2}}\mathbf{y}_\alpha\mathbf{x}_{\alpha+1}q^{-\frac{\mathbf{h}_{\alpha+1}}{2}},\\ F_\alpha &= \frac{1}{(q-q^{-1})}q^{-\frac{\mathbf{h}_{\alpha+1}}{2}}\mathcal{E}_{\alpha+1,\alpha}q^{-\frac{\mathbf{h}_\alpha}{2}} = \frac{1}{(q-q^{-1})}q^{-\frac{\mathbf{h}_{\alpha+1}}{2}}\mathbf{y}_{\alpha+1}\mathbf{x}_\alpha q^{-\frac{\mathbf{h}_\alpha}{2}},\\ H_\alpha &= \mathbf{h}_\alpha - \mathbf{h}_{\alpha+1}, \qquad \alpha = 1, \ldots, n-1. \end{aligned} \tag{50}$$

This correspondence immediately follows from a comparison of (46) with the general form of the fundamental L-operator for the algebra $U_q(\widehat{sl}(n))$ given in [19, 49].

Note that the parameters $\{\mu\}$ enter equation (45) trivially; their only effect reduces to the pre-multiplication of $\mathcal{L}(\lambda)$ by the diagonal matrix $\mathtt{diag}(\mu_1, \mu_2, \ldots, \mu_n)$. This corresponds to the introduction of arbitrary 'horizontal fields' [33], which, obviously, does not affect the integrability of the 2d model related to the L-operator (46).

The general decomposition of (39) has the form

$$\mathbf{L}_{\mathbf{Vb}}(\lambda|\{\mu\}) = \bigoplus_{k=0}^{n} \mathcal{L}^{sl(n)}(\omega_k, \lambda|\{\mu\}), \tag{51}$$

where $\mathcal{L}^{sl(n)}(\omega_k, \lambda|\{\mu\})$ is the L-operator corresponding to the kth fundamental representation π_{ω_k} of $U_q(sl(n))$ (with the horizontal fields $\{\mu\}$); in particular, $\mathcal{L}^{sl(n)}(\omega_1, \lambda|\{\mu\})$ exactly coincides with $\mathcal{L}(\lambda|\{\mu_k\})$ defined by ((45) and (46)). For further references define row-to-row transfer matrices corresponding to the above L-operators for an inhomogeneous periodic chain of the length m,

$$\mathbb{T}_m^{sl(n)}(\omega_k, \{\lambda\}|\{\mu\}) = \mathrm{Tr}_{\pi_{\omega_k}}(\mathcal{L}^{sl(n)}(\omega_k, \lambda_m|\{\mu\})\mathcal{L}^{sl(n)}(\omega_k, \lambda_{m-1}|\{\mu\}) \cdots \mathcal{L}^{sl(n)}(\omega_k, \lambda_1|\{\mu\})), \tag{52}$$

where $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, denotes the set of spectral parameters.

For the Fock representation (27) the element $\mathcal{J}$, defined in (47), has the integer-valued spectrum $J = 0, 1, 2, \ldots, \infty$, where J is just the total occupation number of n different q-oscillators. Thus, the quantum space $\mathcal{F}^{\otimes n}$ can be split into a direct sum

$$\mathcal{F}^{\otimes n} = \bigoplus_{J=0}^{\infty} V_J, \qquad \dim V_J = \frac{(J+n-1)!}{J!(n-1)!}, \tag{53}$$

where the spaces V_J are isomorphic to the representation spaces $\pi_{J\omega_1}$ of the algebra $U_q(sl(n))$ (the Jth symmetric powers of the vector representation π_{ω_1}). As the L-operator (39) commute with the element $\mathcal{J}$, it acts invariantly in each component of (53). Indeed, the L-operators appearing on the right-hand side of (51) split into a direct sum of $U_q(\widehat{sl}(n))$ R-matrices, $\mathcal{R}_{J\omega_1,\omega_k}$, corresponding to all representation $\pi_{J\omega_1}$, $J = 0, 1, 2, \ldots, \infty$ in quantum space. Likewise, operator (36) splits into an infinite direct sum of (appropriately normalized) R-matrices, $\mathcal{R}_{J\omega_1,J'\omega_1}$, corresponding to the representations $\pi_{J\omega_1}$ and $\pi_{J'\omega_1}$

$$\mathbf{R}_{\mathbf{bc}} = \bigoplus_{J=0}^{\infty} \bigoplus_{J'=0}^{\infty} \mathcal{R}_{J\omega_1,J'\omega_1}, \tag{54}$$

where, for brevity, the spectral parameter and field arguments are suppressed. This decomposition can be established directly from definition (36); the calculation of each term on the right-hand side requires the use of the polynomials $P_n(x, y, z)$, recursively defined by (31), up to the order $n = J$ (see [44] for further details).

6. The rank-size duality

Consider now the layer-to-layer transfer matrix, $\mathbf{T}_{mn}$ of m columns and n rows obtained as a trace of a product of nm operators (23),

$$\mathbf{T}_{mn}(\{\lambda\}|\{\mu\}) = \mathrm{Trace}\left[\prod_i^{\curvearrowleft}\left(\prod_j^{\curvearrowright} L_{ij}(\mathbf{v}_{ij}, s_{ij})\right)\right], \tag{55}$$

where the ordered products are defined as

$$\prod_j^{\curvearrowright} f_j \stackrel{\text{def}}{=} f_1 f_2 \cdots f_n, \qquad \prod_i^{\curvearrowleft} g_i \stackrel{\text{def}}{=} g_m g_{m-1} \cdots g_1. \tag{56}$$

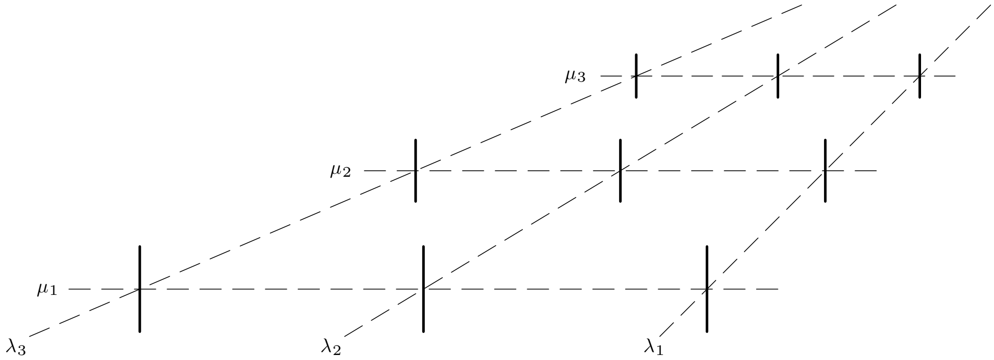

Figure 5. The spacial structure of the layer-to-layer transfer matrix (55).

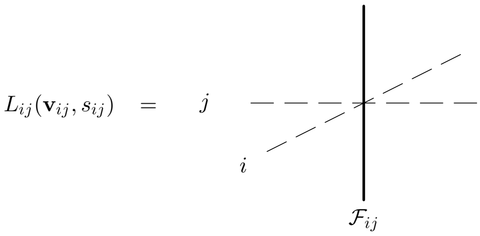

Figure 6. Graphical representation of $L_{ij}(\mathbf{v}_{ij}, s_{ij})$ in (55).

The spacial structure of $\mathbf{T}_{mn}$ is illustrated in figure 5. Each node (i, j) located at the intersection of the ith column and jth row corresponds to the operator $L_{ij}(\mathbf{v}_{ij}, s_{ij})$ in (55), as shown in figure 6. This operator acts in its own quantum space $\mathcal{F}_{ij}$ (these spaces are associated with the vertical lines in figure 5) and in two two-dimensional vector spaces $\mathbb{C}^2$ which are associated with the column and row corresponding to this node. The trace in (55) is taken over all these two-dimensional spaces. Further, each column is assigned with the parameter λ_i, $i = 1, 2, \ldots, m$ and each row is assigned with the parameter μ_j, $j = 1, 2, \ldots, n$. The arguments s_{ij} used in (55) are given by $s_{ij} = (\lambda_i, \mu_j)$.

The layer-to-layer transfer matrices $\mathbf{T}_{mn}(\{\lambda\}|\{\mu\})$ act in the direct product $\mathcal{F}^{\otimes mn}$ of the Fock spaces (27). They form a two-parameter commutative family of operators. Indeed, equation (34) implies that

$$[\mathbf{T}_{mn}(\{\lambda\}|\{\mu\}), \mathbf{T}_{mn}(\{u\lambda\}|\{v\mu\})] = 0, \tag{57}$$

where

$$\{u\lambda\} = \{u\lambda_1, u\lambda_2, \ldots, u\lambda_m\}, \qquad \{v\mu\} = \{v\mu_1, v\mu_2, \ldots, v\mu_n\}, \tag{58}$$

with u and v being arbitrary complex parameters. By construction, the layer-to-layer transfer matrix

$$\mathbf{T}_{mn}(\{u\lambda\}|\{v\mu\}) = \sum_{k=0}^{m} \sum_{\ell=0}^{n} u^{nk} v^{m\ell} \mathbf{G}_{k\ell} \tag{59}$$

is a polynomial in u and v. Its coefficients, $\mathbf{G}_{k\ell}$, provide a set of mutually commuting 'integrals of motion' for this 3d model. The latter non-trivially depend only on the ratios of λ's and the ratios of μ's, since their overall normalizations can be absorbed into the parameters u and v.

The order of the products in definition (55) can be interchanged by using a simple symmetry property,

$$\mathbf{L}_{12}(\mathbf{v}, (\lambda, \mu)) = [\mathbf{L}_{21}(\mathbf{v}, (\mu, \lambda))]^t, \tag{60}$$

where the superscript t denotes the transposition of the four by four matrix (23). In this way, one obtains the following 'reflection' symmetry relation:

$$\mathbf{T}_{mn}(\{\lambda\}|\{\mu\}) = \mathbf{T}_{nm}(\{\mu\}|\{\lambda\}). \tag{61}$$

Substituting now definition (39) for the inner products in two variants of equation (55) (for each side of the last equation), and using decomposition (51) and definitions (52), one arrives at a remarkable duality relation for the transfer matrices of the associated two-dimensional models

$$\sum_{k=0}^{n} v^{km}\mathbb{T}_m^{sl(n)}(\omega_k, \{u\lambda\}|\{\mu\}) = \sum_{\ell=0}^{m} u^{\ell n}\mathbb{T}_n^{sl(m)}(\omega_\ell, \{v\mu\}|\{\lambda\}). \tag{62}$$

The left- and right-hand sides of this relation are just different expansions of the same layer-to-layer transfer matrix (59). It provides two different representations for the same integrals of motion $\mathbf{G}_{k\ell}$ in terms of two-dimensional lattice models related to two different quantum affine algebras $U_q(\widehat{sl}(n))$ and $U_q(\widehat{sl}(m))$. The above relation simultaneously interchanges the rank of the quantum algebra and the set of horizontal fields in one model with the length of the chain and the set of (inhomogeneous) spectral parameters in the other. We will call this relation the 'rank-size' (RS) duality relation. Note that $q \to 1$ limit of this relation covers the 2d solvable models with the Yangian symmetry [18] (the $sl(n)$-invariant Heisenberg magnets and associated models [34–37]).

The zero-field case (i.e., when all λ's and all μ's are equal to one) corresponds to the homogeneous models in both sides of relation (62), with the spectral parameters u and v respectively. For example, when $n = 2$ it relates the (zero-field homogeneous) six-vertex models on the chain of the length m with an $U_q(\widehat{sl}(m))$-type model [30–33] on the chain of just two lattice sites.

It is well known that all two-dimensional models discussed here can be solved by the *nested Bethe Ansatz* [35–37, 52], which allows one to obtain the exact expressions for the eigenvalues and eigenvectors of the transfer matrices in terms of solutions of certain algebraic equations, called the Bethe Ansatz equations. The above RS-duality relation implies a complete equivalence between two different *nested Bethe Ansätze* associated with the two sides of (62). This remarkable (and rather unexpected) equivalence certainly deserves further studies, which are postponed to the future publication [48].

7. Conclusion

In this paper, we have constructed a new solution for the set of the interrelated TE's (3), (4) and (34). These equations involve the two different spaces of states for the discrete edge 'spins': the two-dimensional vector space $\mathbb{C}^2$ and the infinite-dimensional Fock space, $\mathcal{F}$, defined by (27). In particular, the 3d L-operators $L_{ij,a}$, entering equation (4), act in the direct product of the two spaces $\mathbb{C}^2$ and one Fock space $\mathcal{F}_a$. The most important ingredient of our approach is the local Yang–Baxter equation (11) with the specific Ansatz (23) for these 3d L-operators. This equation defines the automorphism (24) of the tensor cube of the q-deformed Heisenberg algebra (13), which almost straightforwardly provides the solution the TE's (3) and (4).

Currently, it is unclear whether the same scheme should work with different 3d L-operators, other than (23) but it is certainly worth exploring this point. Our choice of the form of this operator was motivated by a consideration of the 'resonant three-wave scattering' model [53, 54], which is a well-known example of integrable classical system in 2 + 1 dimensions. This model is described by the nonlinear differential equations,

$$\partial_\alpha A_{\beta\gamma}(x) = A_{\beta\alpha}(x) A_{\alpha\gamma}(x), \qquad (\alpha, \beta, \gamma) = \mathtt{perm}(1, 2, 3), \tag{63}$$

for the fields $A_{\alpha,\beta}(x)$, $\alpha \neq \beta$, in the three-dimensional space with the coordinates $x = (x_1, x_2, x_3)$ and $\partial_\alpha = \partial/\partial x_\alpha$, $\alpha = 1, 2, 3$. They can be thought as the consistency conditions for the following auxiliary linear problem:

$$\partial_\alpha \Psi_\beta(x) = A_{\alpha\beta}(x) \Psi_\alpha(x), \qquad \alpha \neq \beta. \tag{64}$$

The 3d L-operator (23) naturally arises in the quantization [44] of a discrete lattice analogue [42] of this linear problem.

As explained in section 5, every solution of the TE provides an infinite series of quantum R-matrices for 2d solvable lattice models (i.e., the solutions of the Yang–Baxter equation). Our solution of the TE reproduces in this way the R-matrices related to the finite-dimensional highest weight evaluation representations for all the quantum affine algebras $U_q(\widehat{sl}(n))$ with $n = 2, 3, \ldots, \infty$. This 3d interpretation provides a completely new insight into the properties of 2d solvable lattice models, in particular, it leads to a remarkable rank-size (RS) duality, discussed in section 6.

Plausibly, a similar 3d interpretation also exists for the trigonometric R-matrices related to all other infinite series of quantum affine algebras [55, 56] and superalgebras [57]. If so, the corresponding 3d models will, most likely, involve non-trivial boundary conditions [58–60]. We hope to address this interesting problem in the future.

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